

# Fuzzy Rule Interpolation for Multidimensional Input Spaces With Applications: A Case Study

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**Abstract**—Fuzzy rule based systems have been very popular in many engineering applications. However, when generating fuzzy rules from the available information, this may result in a sparse fuzzy rule base. Fuzzy rule interpolation techniques have been established to solve the problems encountered in processing sparse fuzzy rule bases. In most engineering applications, the use of more than one input variable is common, however, the majority of the fuzzy rule interpolation techniques only present detailed analysis to one input variable case. This paper investigates characteristics of two selected fuzzy rule interpolation techniques for multidimensional input spaces and proposes an improved fuzzy rule interpolation technique to handle multidimensional input spaces. The three methods are compared by means of application examples in the field of petroleum engineering and mineral processing. The results show that the proposed fuzzy rule interpolation technique for multidimensional input spaces can be used in engineering applications.

**Index Terms**—Fuzzy rule interpolation, multidimensional input spaces, sparse fuzzy rules.

## I. INTRODUCTION

WHEN fuzzy systems are applied to typical engineering problems, the fuzzy rule base is constructed using any available information. The information can be in the form of measured data or might come from some computational simulation. The fuzzy systems thus created are normally based on classical inference techniques of fuzzy control originally proposed by Zadeh [1], Mamdani [2], and Takagi and Sugeno [3]. Fuzzy models based on these theories have been producing promising results in control applications with the limitation of not having more than five to ten state variables. However, in many cases, the information provided is not enough to construct a complete and comprehensive fuzzy rule base or, its complexity does not allow a tractable approach; these problems require different techniques of reasoning.

In the case when a fuzzy rule base contains gaps, which is called sparse rule base, classical fuzzy reasoning methods can no longer be used. This fact is due to the lack of traditional inference mechanism in the case when observations find no fuzzy rule to

fire. This cannot be allowed when using a fuzzy system in any engineering application and such a fuzzy system is considered useless. Containing gaps could be a major drawback from using fuzzy systems in many engineering applications. Fuzzy rule interpolation techniques provide a tool for specifying an output fuzzy set even when one or all of the input spaces are sparse. Kóczy and Hirota (KH) [4] introduced the first interpolation approach known as (linear) KH interpolation (named after its inventors), which was the basis of many other interpolation techniques described in Section II. These methods determine the conclusion by its  $\alpha$ -cuts in such a way that the ratio of distances (in some sense) between the conclusion and the consequents should be identical with that among observation and the antecedents for all important  $\alpha$ -cuts, e.g., breakpoint levels. In most fuzzy applications, the input vector involves more than one variable, therefore the characteristics of fuzzy rule interpolation for multidimensional input spaces is of much interest [5], [6]. This paper presents a new technique to perform fuzzy rule interpolation for multidimensional input spaces referred to as IMUL.

The paper is organized as follows. Section II gives an overview of fuzzy rule interpolation techniques. Section III presents the extension of the basic fuzzy rule interpolation theories to be used for multidimensional input spaces. Section III finishes with the presentation of the new technique. Section IV presents results of the new technique. Section IV-A presents the results of IMUL as compared with results of MACI and KH methods; the latter two are the most used fuzzy rule interpolation techniques. Section IV-B presents application of these three methods to petroleum engineering and Section IV-C presents results with an application to mineral engineering. Finally, some concluding remarks are stated in Section V.

## II. OVERVIEW OF FUZZY RULE INTERPOLATION TECHNIQUES

In this section, we give an overview of fuzzy rule interpolation techniques based on [7], with special emphasis on the first published (and most commonly used)  $\alpha$ -cut-based fuzzy interpolation, termed KH interpolation (Section II-C).

### A. Fuzzy Rule Based Interpolation Techniques

The classical inference methods in fuzzy control (Zadeh, Mamdani and Sugeno) deal with dense rule bases, where the input space is completely covered by the rule premises. If the universe of discourse is not completely covered by the rule antecedents it can happen that for certain observation no rule is fired. Fuzzy rule interpolation, proposed first by Kóczy and Hirota [4], is an inference technique for fuzzy rule bases, whenever the antecedents do not cover the whole input universe, i.e.,

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for so-called sparse fuzzy rule bases. Let us now consider the reasons that could lead to sparse or incomplete rule bases.

- 1) Fuzzy inference methods are often criticized when the number of inputs is large (over 10): the size of the rule base and the complexity of the inference algorithm grow exponentially with the number of inputs. A possible solution to break down the complexity is the omission of redundant rules. This can, however, lead to incomplete rule bases [8].
- 2) Incomplete knowledge about the modeled system—regardless of the technique of construction—can result in sparse rule bases. At the beginning, on the basis of Zadeh's initial concept about linguistic variables, fuzzy systems were constructed from linguistic IF-THEN rules provided by human experts. More recently, learning techniques have increasingly been developed and applied to the construction of fuzzy IF-THEN rules from numerical sample data. Both methods of construction can result in sparse rule bases. In the former case, an incomplete rule base can be the consequence of missing expertise for certain system configurations and state space regions. In the latter case, it may happen that sample data do not represent sufficiently some regions of the input domain.
- 3) Even originally dense rule bases (inputs are completely covered by rule premises) can turn to be sparse: by tuning the rules of a dense rule base, rule premises can be partially shifted and shrunk in such a way that the tuned model will also contain gaps [9]. The reason of this phenomenon is that tuning by learning is usually concentrates to typical, frequently occurring situations, which are represented by the majority of training data/expert knowledge.
- 4) "Gaps" can be defined between rule bases intentionally, in order to avoid too high complexity in very large systems. Hence, fuzzy interpolation techniques have important role in hierarchically structured systems [10].

The rule bases containing gaps require completely new techniques of reasoning and control as compared to the original Zadeh (CRI), Mamdani (Larsen, etc.), and TS-algorithms. The family of methods works well only if the system has the "nice" property of not behaving too unexpectedly in areas where the model does not cover. Luckily, in practice such a nice behavior might be expected in most cases. The class of systems where the following algorithms can be applied termed "interpolative system."

### B. A Survey of Interpolation Techniques

The first result published in this field was given by Kóczy and Hirota [4]. It is applicable to convex and normal fuzzy (CNF) sets. It determines the conclusion by its  $\alpha$ -cuts in such a way that the ratio of distances between the conclusion and the consequents should be identical with the ones between the observation and the antecedents for all important  $\alpha$ -cuts. The fundamental equation of rule interpolation [FERI, see also Section II-C, (2)] describes the connection between the ratios of distances for  $\alpha$ -cuts. FERI is in accordance with the gradual semantic interpretation, proposed first by Dubois and Prade in 1992 [11], "the more similar is the observation to an antecedent the more similar the conclusion should be to the consequent corresponding to the given an-

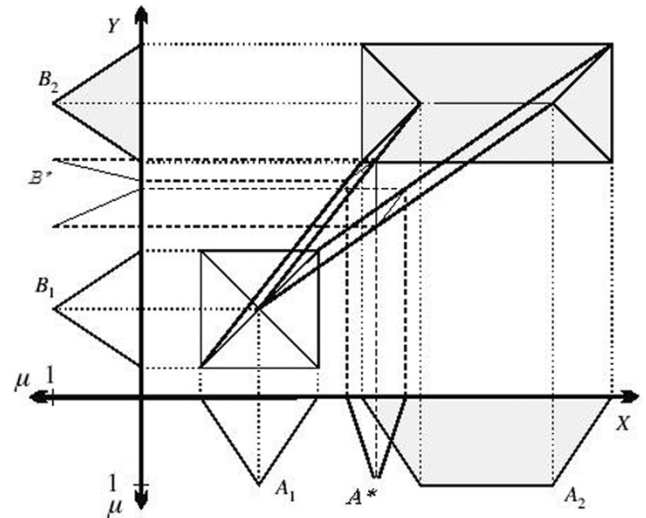


Fig. 1. Abnormal conclusion generated by the KH technique.

tecedent." This semantic rule interpretation is an improvement of the approximate analogical reasoning technique proposed by Türkşen [12] "the closer the observation is to an antecedent the closer the conclusion should be to the corresponding consequent of the given antecedent." In this work, Türkşen defined the distance between the sets by the measure of overlapping [13] after a very thorough overview and evaluation of various definitions of distance between fuzzy sets. The Revision Principle developed by Ding *et al.* in 1993, which constructs the conclusion by means of a so-called semantic curve [14]–[16], can be considered as another antecedent of rule interpolation.

The main purpose to introduce fuzzy rule interpolation was the great computational complexity requirement of classical fuzzy reasoning methods [8], [17]. Rule interpolation is efficient if the shape of the rules is simple, most frequently piecewise linear, even, triangular or trapezoidal, since in these cases the rules, i.e., the fuzzy sets involved in them, can be described with only a few characteristic points. It is a natural demand that the method should determine the conclusion based only on a sufficient number of  $\alpha$ -cuts, namely, based on the characteristic points (or breakpoint levels) of the involved sets, because otherwise the calculation becomes too "expensive". Although it could be expected that the conclusion preserves the linearity of the premises, it is not satisfied in general, i.e., the shape of the conclusion can be different from the shape of the other involved sets. Kóczy and Kovács [18], [19], Kawase and Chen [20], and Shi and Mizumoto [21]–[23] examined the condition for preserving linearity for the generated conclusion. The investigations give estimates for the linear deviation error, which turns out to be considerably low for most practical cases. This means that it is sufficient to calculate the conclusion only for characteristic points.

Another problematic point of the original method that stimulated researchers to modify the original KH rule interpolation approach is the following. The KH interpolation technique, though it shows advantageous properties in several aspects, has the shortcoming that under certain configurations of the input fuzzy sets (too heterogeneous shape) the calculated conclusion is abnormal, i.e., not directly interpretable or even completely empty as the fuzzy set obtained as conclusion shown in Fig. 1.

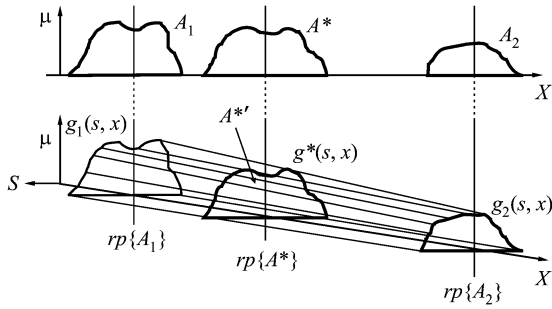


Fig. 2. Formation of solid in the input dimension and determination of  $A^{*'}.$

Now we give a brief overview of the alternative fuzzy rule interpolation techniques proposed to solve this problem.

The method proposed by Vass *et al.* [4], [8], [24] decreases the applicability limit of the method, but does not eliminate it completely. Authors compute the conclusion based on the distance of the central points and the supports of fuzzy sets.

Conceptually different approaches were proposed by Baranyi *et al.* [25]–[28] based on the relation of the fuzzy sets and by Baranyi *et al.* in 1998 [29] based on the semantic and inter-relational features of fuzzy sets. They determined the location (central point or most typical point) of the conclusion based on the ratio of the centers of the observation and the antecedents. After all involved sets are rotated by  $90^\circ$  around their centres, and by connecting the corresponding points of antecedents and consequents, two solids can be formed: one in the input and one in the output dimension. On Fig. 2 we depicted the solid formed in the input dimension. The solids are cut at the centres of the observation and at the determined location of the conclusion, respectively, which results in the set  $A^{*'}$  in the input space and in the set  $B^{*'}$  in the output space. Then, a revision function is used to determine the final conclusion  $B^*$  based on the similarity of the observation  $A^*$  and the “interpolated” observation  $A^{*'}$ . A practically applicable technique for determining the exact shape of this solid is described in detail in [28].

These methods have numerous advantages, such as

- they always give an interpretable conclusion as a “real” fuzzy set, i.e., any abnormal shape of the conclusion is precluded;
- they can be applied to arbitrary shaped fuzzy sets, i.e., neither convexity nor normality is required, only the centres of the sets are supposed to be ordered. It means that some part of the observation can even exceed the support of antecedents;
- versions specialized for piecewise linear fuzzy sets produce piecewise linear fuzzy set as conclusions, hence the shape of the sets at hand is preserved.

The only problematic point of these methods is that the calculation of the revision function even for the special piecewise linear case needs considerable time, thus one of the most important reasons for inventing fuzzy interpolation techniques is violated or at least partly neglected.

Another fuzzy interpolation technique was developed in 1996 by Gedeon and Kóczy [30] founded on the preservation of “relative fuzziness,” a term referring to the size a fuzzy set’s flanks (see also Fig. 5). This approach cannot be applied if any of the

consequent sets are crisp, because of a zero divisor in the formulae. This technique was extended by Kóczy in 1997 [31], which was suitable for the above mentioned crisp sets, as well. The authors also showed its immediate connection with FERI. These techniques are applicable also to CNF sets.

In 1996, Kovács and Kóczy proposed yet another interpolation technique based on the approximation of the vague environment of fuzzy rule bases [32]–[34].

A further modified variant of the original technique was published by Tikk and Baranyi in [7], which was termed modified alpha-cut based interpolation (MACI). MACI technique solves the abnormality problem effectively, while it maintains the advantageous properties of the original approach. In [35] the authors show that MACI can be applied to arbitrary shaped fuzzy sets, i.e., even the normality of sets can be relaxed. Obviously, the complexity of the involved fuzzy sets increases the calculation need required to determine the conclusion.

We remark that recently Jenei gave an axiomatic characterization of fuzzy rule interpolation [36]–[38]. This work is however more theoretical and does not propose any easily implementable algorithm, although it completely excludes abnormality in the conclusion.

### C. The KH Rule Interpolation Technique

Let us introduce the concept of  $\alpha$ -cut-distance-based KH rule interpolation. Every fuzzy sets can be approximated with the use of the family of its  $\alpha$ -cuts. Theoretically all infinite many cuts should be calculated for the approximation that would yield a combinatorial explosion. In most practical cases, however, if the membership function is piecewise linear, it is often sufficient to calculate its  $\alpha$ -cuts for only a few important or typical  $\alpha$  values [18], [21], [22], e.g., in the trapezoidal or triangular cases for  $\alpha = 0$  and  $\alpha = 1$ .

The KH rule interpolation algorithm requires the following conditions to be fulfilled: the fuzzy sets in both premises and consequences have to be CNF sets with bounded support, having also continuous membership functions. When input and output universes are bounded and gradual—guaranteeing the existence of a total ordering on them—a partial ordering can be introduced among CNF sets of the input by means of their  $\alpha$ -cuts

$$\forall \alpha \in [0, 1] : \inf\{A_\alpha\} \leq \inf\{B_\alpha\} \\ \text{and } \sup\{A_\alpha\} \leq \sup\{B_\alpha\} \implies A \prec B$$

i.e., in words,  $A$  precedes  $B$ . Here,  $F(X)$  denotes the set of all CNF sets of  $X$ .

Based on comparable fuzzy sets, the concept of fuzzy distance can be introduced, and this reduces the problem of determining conclusion in sparse rule bases to the application of classical function approximation techniques like interpolation or extrapolation (more details in [39]).

The simplest of these techniques is the linear interpolation of two rules for the area between their antecedents. This can be applied if the observation  $A$  is located so that

$$A_1 \prec A^* \prec A_2 \text{ and } B_1 \prec B_2 \quad (1)$$

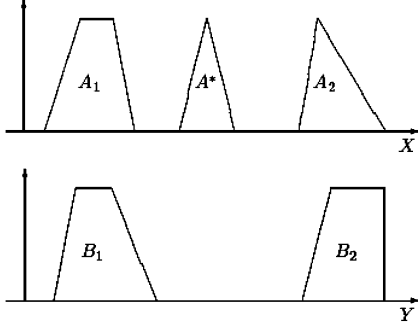


Fig. 3. Location of involved fuzzy sets.

with an apparent ordering  $\prec$ . Here  $A_1 \rightarrow B_1$  and  $A_2 \rightarrow B_2$  form the pair of flanking rules for the observation  $A^*$  (see Fig. 3).

Using the concept of fuzzy distance  $d : F(X) \times F(X) \mapsto R$ , FERI can be written as

$$d(A^*, A_1) : d(A^*, A_2) = d(B^*, B_1) : d(B^*, B_2) \quad (2)$$

where  $B^*$  (the consequent) is unknown. After decomposing (2) for all  $\alpha \in [0, 1]$  (in practice only breakpoint levels are considered), it can be solved for every  $B_\alpha^*$  independently.

The following formulas are the solution for the linear KH interpolator ( $L$  and  $U$  denote “lower” and “upper” fuzzy distance, respectively)

$$\begin{aligned} \min\{B_\alpha^*\} &= \frac{\frac{\inf\{B_{1\alpha}\}}{d_L(A_\alpha^*, A_{1\alpha})} + \frac{\inf\{B_{2\alpha}\}}{d_L(A_\alpha^*, A_{2\alpha})}}{\frac{1}{d_L(A_\alpha^*, A_{1\alpha})} + \frac{1}{d_L(A_\alpha^*, A_{2\alpha})}} \\ \max\{B_\alpha^*\} &= \frac{\frac{\sup\{B_{1\alpha}\}}{d_U(A_\alpha^*, A_{1\alpha})} + \frac{\sup\{B_{2\alpha}\}}{d_U(A_\alpha^*, A_{2\alpha})}}{\frac{1}{d_U(A_\alpha^*, A_{1\alpha})} + \frac{1}{d_U(A_\alpha^*, A_{2\alpha})}}. \end{aligned} \quad (3)$$

The two families of solutions determine a fuzzy set  $B^*$ , if they satisfy  $\min\{B_\alpha^*\} \leq \max\{B_\alpha^*\}$  for every  $\alpha$ , cf. [39], and also  $\min\{B_{\alpha_1}^*\} \leq \min\{B_{\alpha_2}^*\}$  and  $\max\{B_{\alpha_1}^*\} \geq \max\{B_{\alpha_2}^*\}$  whenever  $\alpha_1 \geq \alpha_2$ . In certain cases these conditions are not satisfied and, hence, we obtain a conclusion that is not directly interpretable as a fuzzy set or that is completely empty. This is the most important disadvantage of the linear KH technique apart from the restriction on the shape of the input sets, which is rather intuitive for most real applications.

In order to alleviate this problem, conditions were imposed by Kóczy and Kovács [18], Kawase and Chen [20], and Shi and Mizumoto [21], [23] so as to yield a real fuzzy set (see also the survey on other proposed approaches in the previous Section II-B).

The principle of interpolating two rules can be extended in many different ways. One of the most obvious extensions of the interpolation of two rules is the interpolation of  $2q$  rules ( $q$  and  $q$  flanking the observation in the sense of  $\prec$ ), where pairs of flanking rules are considered, and the further the elements of the pair from the observation are located, the less weight the respective consequents play in the construction of the conclusion. It is obtained from the solution of (3) repeatedly for the pairs

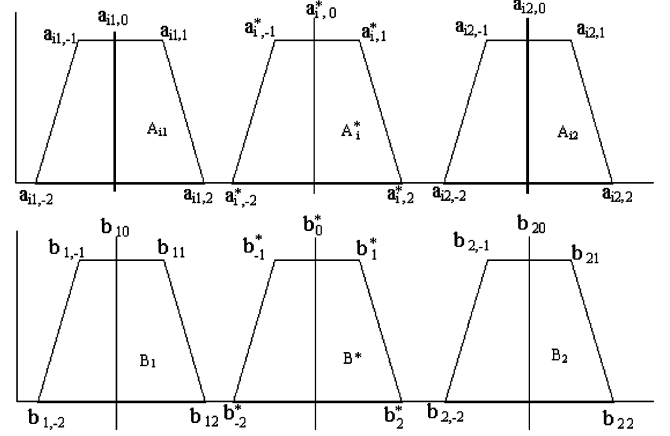


Fig. 4. Notations used.

of points and by averaging the various solutions in a weighted way. The overall solutions are

$$\begin{aligned} \min B_\alpha^* &= \frac{\sum_{i=1}^{2q} \frac{1}{d_L(A_\alpha^*, A_{i\alpha})} \inf\{B_{i\alpha}\}}{\sum_{i=1}^{2q} \frac{1}{d_L(A_\alpha^*, A_{i\alpha})}} \\ \max B_\alpha^* &= \frac{\sum_{i=1}^{2q} \frac{1}{d_U(A_\alpha^*, A_{i\alpha})} \sup\{B_{i\alpha}\}}{\sum_{i=1}^{2q} \frac{1}{d_U(A_\alpha^*, A_{i\alpha})}}. \end{aligned} \quad (4)$$

More details on this technique can be found in [4]. It was shown in [40] that the generalized form of KH interpolator is a mathematically stable technique, i.e., independently of the location of the fuzzy sets being the basis of the calculation, the KH interpolator is able to approximate “well” any continuous function; i.e., KH interpolator possesses a practically enhanced version of the general universal approximator property.

### III. FUZZY RULE INTERPOLATION FOR MULTIDIMENSIONAL INPUT SPACES

In real fuzzy applications, the input vector involves more than one variable, therefore the characteristics of fuzzy rule interpolation for multidimensional input spaces is of much interest. In fact, the advantages of using sparse rule bases manifest only with a few to a few dozen variables really. In this paper, we will limit ourselves to the analysis of only three techniques that can be extended for use in multidimensional input spaces: the original KH fuzzy interpolation technique [4], the modified  $\alpha$ -cut fuzzy interpolation (MACI) technique [7] and finally the new improved fuzzy interpolation technique for multidimensional input spaces (IMUL) proposed here. Fig. 4 shows the notations for two trapezoidal rules used in the following analysis.

#### A. KH Fuzzy Rule Interpolation for Multidimensional Input Spaces

The KH fuzzy interpolation can be extended to multidimensional input spaces, after normalization of the dimensions by dividing the values in  $X_i$  by  $\min\{X_i\}$ ,  $\max\{X_i\}$  for every  $i$ , thus obtaining  $X_{\text{norm}} = X_{1\text{norm}} \times X_{2\text{norm}} \times \dots \times X_{k\text{norm}} = [0, 1]^k$ ,

by using the Euclidean distance on all input spaces. For  $k$  input dimensions, the following hold.

The right core point for trapezoidal membership

$$b_1^* = \frac{d_{1RC} * b_{11} + d_{2RC} * b_{21}}{d_{1RC} + d_{2RC}} \quad (5)$$

where

$$d_{1RC} = \sqrt{\sum_{i=1}^k (a_{i2,1} - a_{i^*,1})^2}$$

and

$$d_{2RC} = \sqrt{\sum_{i=1}^k (a_{i^*,1} - a_{i1,1})^2}.$$

The right flank support point is defined by the expression,

$$b_1^* = \frac{d_{1RF} * b_{12} + d_{2RF} * b_{22}}{d_{1RF} + d_{2RF}} \quad (6)$$

where

$$d_{1RF} = \sqrt{\sum_{i=1}^k (a_{i2,2} - a_{i^*,2})^2}$$

and

$$d_{2RF} = \sqrt{\sum_{i=1}^k (a_{i^*,2} - a_{i1,2})^2}.$$

The left flank support and left core points can be calculated in a similar way.

### B. MACI Fuzzy Rule Interpolation for Multidimensional Input Spaces

MACI works with the vector description of fuzzy sets. The fuzzy set  $A$  is represented by a vector  $a = [a_{-m}, \dots, a_0, \dots, a_n]$ , where  $a_k$  ( $k \in [-m, n]$ ) are the characteristic points of  $A$ , and  $a_0$  is the reference point of  $A$  with membership degree one. This means that  $a_L = [a_{-m}, \dots, a_0]$ , and  $a_R = [a_0, \dots, a_n]$  are the left flank and right flank of  $A$ , respectively. Similarly as in the case of KH interpolation, the basic technique of MACI is extended to multidimensional input spaces using Euclidean distance on all input spaces. In our case, the reference points of all membership functions can be calculated by taking the mid point of the membership function. For  $k$  input dimensions, the reference point of the interpolated conclusion for trapezoidal membership function is

$$b_0^* = (1 - \lambda_{\text{core}})b_{10} + \lambda_{\text{core}}b_{20} \quad \text{where} \quad \lambda_{\text{core}} = \frac{\sqrt{\sum_{i=1}^k (a_{i^*,0}^* - a_{i1,0})^2}}{\sqrt{\sum_{i=1}^k (a_{i2,0} - a_{i1,0})^2}}. \quad (7)$$

With the reference point the left and right cores can be calculated. For the right core

$$b_0^* = (1 - \lambda_{\text{right}})b_{11} + \lambda_{\text{right}}b_{21} + (\lambda_{\text{core}} - \lambda_{\text{right}})(b_{20} - b_{10}) \quad (8)$$

where

$$\lambda_{\text{right}} = \frac{\sqrt{\sum_{i=1}^k (a_{i^*,1}^* - a_{i1,1})^2}}{\sqrt{\sum_{i=1}^k (a_{i2,1} - a_{i1,1})^2}}.$$

For the right flank

$$b_1^* = (1 - \lambda_{\text{rightflk}})b_{12} + \lambda_{\text{rightflk}}b_{22} + (\lambda_{\text{core}} - \lambda_{\text{right}})(b_{20} - b_{10}) + (\lambda_{\text{right}} - \lambda_{\text{rightflk}})(b_{21} - b_{11}) \quad (9)$$

where

$$\lambda_{\text{rightflk}} = \frac{\sqrt{\sum_{i=1}^k (a_{i^*,2}^* - a_{i1,2})^2}}{\sqrt{\sum_{i=1}^k (a_{i2,2} - a_{i1,2})^2}}.$$

The left side is calculated in a similar way.

MACI will only yield singleton conclusion if and only if the consequents are singletons themselves.

### C. IMUL Fuzzy Rule Interpolation for Multidimensional Input Spaces

This technique incorporates certain features of the MACI and the conservation of fuzziness technique, which is also based on FERI in the original KH interpolation sense [30]. IMUL uses the vector representation form of fuzzy sets (first proposed in [41]), and it applies the coordinate transformation features of MACI. At the same time, it can take the fuzziness of the fuzzy sets in the input space at the conclusion as those are presented in the conservation of fuzziness technique. The advantage of this fuzzy interpolation technique is not only that it takes the fuzziness of the sets at the input spaces, but also takes the core at the consequents into the calculation.

For  $k$  input dimensions, the reference characteristic point of the interpolated conclusion with the use of Euclidean distance is:

$$b_0^* = (1 - \lambda_{\text{core}})b_{10} + \lambda_{\text{core}}b_{20} \quad \text{where} \quad \lambda_{\text{core}} = \frac{\sqrt{\sum_{i=1}^k (a_{i^*,0}^* - a_{i1,0})^2}}{\sqrt{\sum_{i=1}^k (a_{i2,0} - a_{i1,0})^2}}. \quad (10)$$

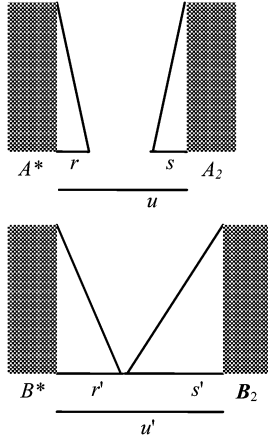


Fig. 5. Ratio for the fuzziness.

By using the previous reference point, the right core of the conclusion calculated as

$$b_1^* = (1 - \lambda_{\text{right}})b_{1,1} + \lambda_{\text{right}}b_{2,1} + (\lambda_{\text{core}} - \lambda_{\text{right}})(b_{20} - b_{10}) \quad (11)$$

where

$$\lambda_{\text{right}} = \frac{\sqrt{\sum_{i=1}^k (a_{i,1}^* - a_{i1,1})^2}}{\sqrt{\sum_{i=1}^k (a_{i2,1} - a_{i1,1})^2}}.$$

The left core is obtained analogously.

After calculating the two sides of the core, the two flanks can be determined. When calculating the left and right flanks of the conclusion, the relative fuzziness of the fuzzy sets in all the input spaces is taken into consideration as follows. See Fig. 5.

First, we present the calculation of the right flank of the conclusion. Based on  $A_{i2}$  and  $B_2$ , the fuzziness of the antecedents ( $s_i$ ) and consequent ( $s'$ ) can be calculated as

$$s_i = a_{i2,-2} - a_{i2,-1} \quad (12)$$

$$s' = b_{2,-2} - b_{2,-1}. \quad (13)$$

The fuzziness of the observed antecedents ( $r_i$ ) and the interpolated consequent ( $r'$ ) can then be obtained as

$$r_i = a_{i,2}^* - a_{i,1}^* \quad (14)$$

$$r' = b_2^* - b_1^*. \quad (15)$$

The distance of the antecedents and consequent are calculated as follows:

$$u_i = a_{i2,0} - a_{i,0}^* \quad (16)$$

$$u' = b_{20} - b_0^*. \quad (17)$$

In multidimensional input spaces

$$s = \sqrt{\sum_{i=1}^k (s_i)^2} \quad (18)$$

$$r = \sqrt{\sum_{i=1}^k (r_i)^2} \quad (19)$$

$$u = \sqrt{\sum_{i=1}^k (u_i)^2}. \quad (20)$$

For the right flank

$$b_2^* = b_1^* + r \left( 1 + \left| \frac{s'}{u'} - \frac{s}{u} \right| \right). \quad (21)$$

The technique is applicable if the observations in each dimension  $i (1, \dots, k)$  fulfils

$$A_{i1} \prec_i A_i^* \prec_i A_{i2} \quad (22)$$

in the sense that the corresponding coordinates in the vector representation of these fuzzy sets are monotone according to  $\prec_i$ , the ordering in the  $i$ th dimension.

The following statement follows from the properties of the MACI, but as in [7] it is proved only for the one-dimensional case, it is extended here.

**Proposition 1:**  $\lambda_{\text{core}}, \lambda_{\text{left}}, \lambda_{\text{right}} \in [0, 1]$  if in all dimensions  $(1, \dots, k)$  the location of the antecedents fulfil expression (22).

*Proof:* We show this for  $\lambda_{\text{core}}$ . The proof for the other two parameters is then straightforward. For  $\lambda_{\text{core}}$  it is obvious because condition (22) implies  $a_{i,0}^* - a_{i1,0} \leq a_{i2,0} - a_{i1,0}$  as  $a_{i,0}^* \leq a_{i2,0}$  for every  $i = 1, \dots, k$ . The marginal cases are obtained when the reference points of all the observations are identical with the reference points of either the left or the right antecedents.  $\square$

**Theorem 1:** With IMUL technique, the slopes of the consequent never collapse, i.e.,

$$b_{-2}^* \leq b_{-1}^* \leq b_0^* \leq b_1^* \leq b_2^*. \quad (23)$$

*Proof:* We prove the theorem for the right flank. The results for the left flank can be obtained similarly.

First, we show that  $b_0^* \leq b_1^*$ , which is, in fact, a consequence of the MACI algorithm used

$$\begin{aligned} b_1^* - b_0^* &= (1 - \lambda_{\text{right}})b_{11} + \lambda_{\text{right}}b_{21} \\ &\quad + (\lambda_{\text{core}} - \lambda_{\text{right}})(b_{20} - b_{10}) \\ &\quad - (1 - \lambda_{\text{core}})b_{10} - \lambda_{\text{core}}b_{20} \\ &= (1 - \lambda_{\text{right}})(b_{11} - b_{10}) + \lambda_{\text{right}}(b_{21} - b_{20}) \\ &\geq 0. \end{aligned} \quad (24)$$

The last inequality holds, because Proposition 1 ensures that  $\lambda_{\text{right}}$  is in the unit interval and (1) implies that the second members of the products cannot be negative.

Second, we show that  $b_1^* \leq b_2^*$ . This is straightforward from the definition of  $b_2^*$ , as  $r$  is nonnegative and  $(1 + |s'/u' - s/u|)$  is positive.  $\square$

**Proposition 2:** The IMUL technique results in singleton fuzzy set if and only if all the observations are crisp (in the

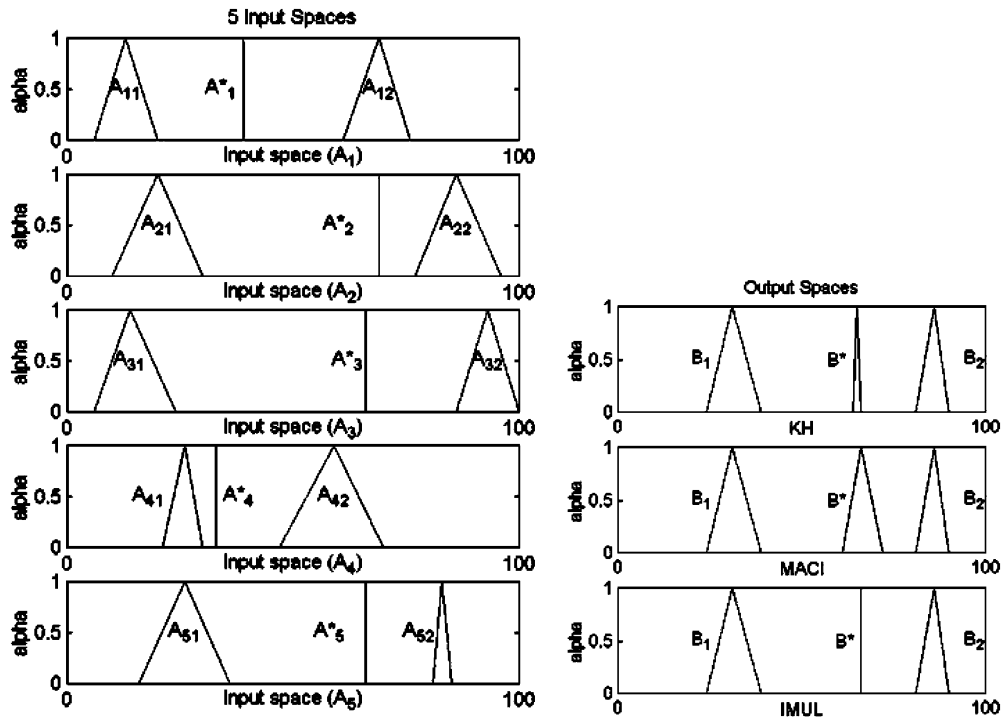


Fig. 6. Singleton fuzzy set.

sense that apart from the core, the membership degrees vanish) and the core of the consequents is single point.<sup>1</sup>

*Proof:*  $b_1^* = b_0^*$ : This is the consequence of the second condition, as then in (24) both tags vanish.

$b_2^* = b_1^*$ : It follows from the first condition, because from this  $r_i = a_{i,-1}^* - a_{i,-2}^* = 0$  for all  $i \in [1, k]$  and, thus,  $r = 0$ .  $\square$

*Remark:* It is a very advantageous property of IMUL that no defuzzification is needed in the case of crisp observations. The proper selection of defuzzification method is usually a pretty difficult issue for system designers, because it is problem dependent. This problem can be avoided by using IMUL, if it can be ensured that all observations are crisp.

#### IV. CASE STUDY AND DISCUSSIONS

In this section, we will first take a look at the results generated by the three fuzzy rule interpolation techniques (KH, MACI, and IMUL) discussed in the previous section. After their results have been shown and compared, we will further examine the possible use of these techniques in two real world applications: well log analysis in petroleum engineering and hydrocyclone control system in mineral processing.

##### A. Comparison Study

In this first test, a total of five input dimensions are used to predict one output dimension. IMUL fuzzy rule interpolation technique will result in a singleton fuzzy set if and only if all the observations are crisp and the core of the consequent is only one point as shown in Fig. 6.

From Fig. 6, it can be concluded that all three techniques generate similar results. Observe that IMUL has the advantage of obtaining the prediction results directly from the interpolated results, i.e., as remarked earlier, no defuzzification is needed.

<sup>1</sup>The consequents are so-called fuzzy numbers.

In the second test, we use three input dimensions to show that IMUL and MACI could avoid the undesirable feature of the original KH interpolation. The result can be observed from Fig. 7.

In the third test, we use two input dimensions to show that IMUL has another advantage feature over KH and MACI such that the interpolated results inherit the fuzziness from the input space rather than the output space. In other words, the interpolated  $B^*$  should be similar to  $A^*$  rather than  $B_1$  and  $B_2$ . Fig. 8 shows the results of this advantage feature of IMUL over KH and MACI. We can interpret this feature in such a way that IMULs result is more accurate, as the conclusion is less fuzzy.

In these case studies, we have shown that IMUL can be used to perform fuzzy rule interpolation for multidimensional input spaces, with advantages over KH and MACI fuzzy interpolation technique. In Sections IV-B and C, we will show the use of the three methods for two real world applications.

##### B. Application to Well Log Analysis

In petroleum reservoir modeling, boreholes are drilled at different locations around the region. Well logging instruments are lowered into the borehole to collect data at different depths known as *well log data*. Well logging instruments used in the measurement of well log data fall broadly into three categories: electrical, nuclear, and acoustic [42]. Examples are gamma ray (GR), resistivity (RT), spontaneous potential (SP), neutron density (NPHI), and sonic interval transit time (DT). There are over fifty different types of logging tools available for different requirements. Beside the well log data, samples from various depths are also obtained and undergo extensive laboratory analysis. This laboratory analysis data is known as *core data*. In well log analysis, the objective is to establish an accurate interpretation model for the prediction of petrophysical characteristics such as porosity, permeability and volume of clay for uncured

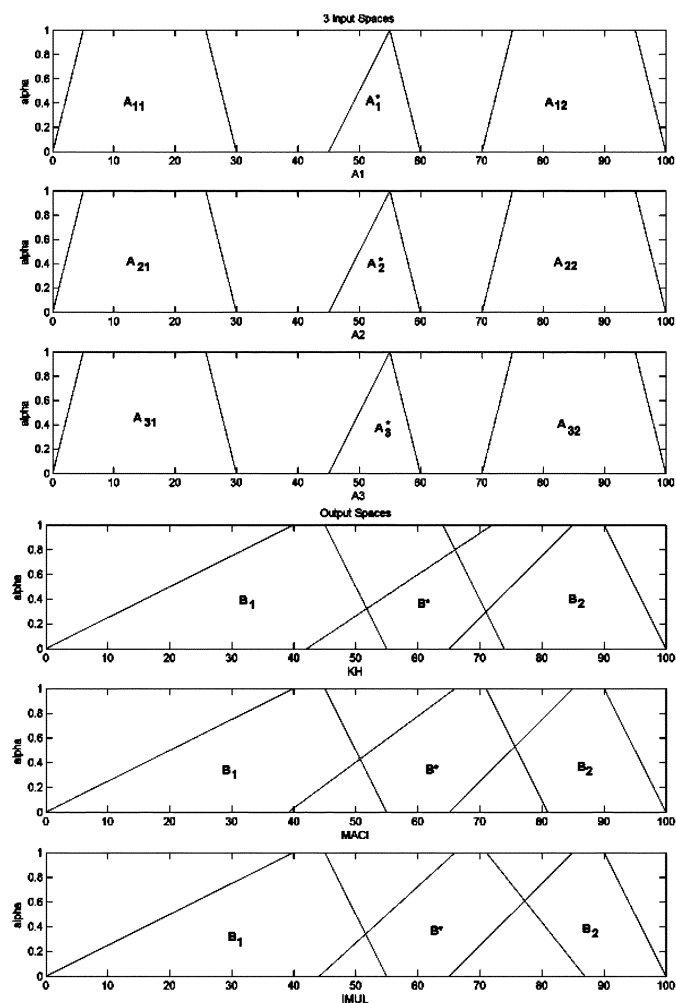


Fig. 7. Undesirable feature of KH interpolation.

depths and boreholes around the region [43], [44]. Such information is essential to the determination of the economic viability of a reservoir to be explored.

Normally the information embedded in available core data is not enough to cover the whole range of values. With the use of fuzzy rule extraction techniques, fuzzy rules generated from these core data form a sparse fuzzy rule base, so conventional fuzzy reasoning techniques cannot be used here. This is due to the lack of an inference mechanism in cases when observations find no fuzzy rule to fire, in uncored depths or wells around the region. This is undesirable when using a fuzzy interpretation model. If more than half the input instances in the prediction well cannot find any rule to fire, this interpretation model is considered useless.

In this case study, data from two wells in the same region are used. The input well logs used are GR, deep induction resistivity (ILD), and DT. They are used to predict the petrophysical property porosity (PHI). Core data from one well are used to establish a prediction model based on the fuzzy rule extraction algorithm. The model is then used to predict the porosity in the second well. All variables are normalized between the values of 0 and 1. The first well has a total of 71 core data and is used to establish the fuzzy rules. The second well has 51 core data and is used as the testing well to determine the prediction accuracy. As the fuzzy rule extraction algorithm is not the main focus of

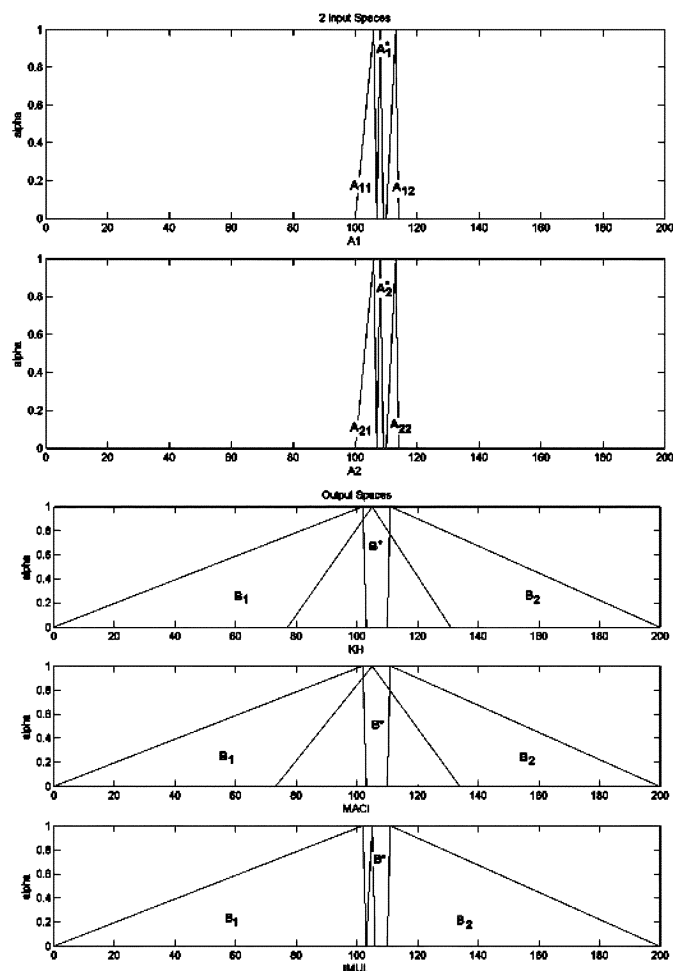


Fig. 8. Broad interpolated fuzzy memberships.

this paper, we will not discuss the details here. The self-generating fuzzy rules algorithm that has shown successful application in this field is used [45]. After all the fuzzy rules have been set up, the input instances from the second well are used to infer the predicted PHI. Using Mamdani type fuzzy inference system [2], it was found that two input instances could not find any rule to fire. When no fuzzy rule can be found to produce a reasonable inference, they are set to zero by default.

After the two input instances have been picked up that do not have any fuzzy rule to fire, the nearest fuzzy rules in the established fuzzy rule base need to be selected. From observation and Euclidean distance measured on each input variable, the nearest fuzzy rules of the two input instances are determined for use by IMUL. We proceed analogously when applying KH and MACI methods. IMUL technique mentioned in the previous section is used as a fuzzy rule interpolation technique to interpolate the inference results of the input instances that find no fuzzy rules to fire. All values have been normalized between 0 and 100 when performing fuzzy rule interpolation.

In order to highlight the significance of using IMUL fuzzy rule interpolation in real world application, we have designed two tests. In the first test, we applied input instances to a fuzzy inference system that showed that two input instances could not find any fuzzy rules to perform inference and set to zero by default. After which, the predicted PHI from this fuzzy inference



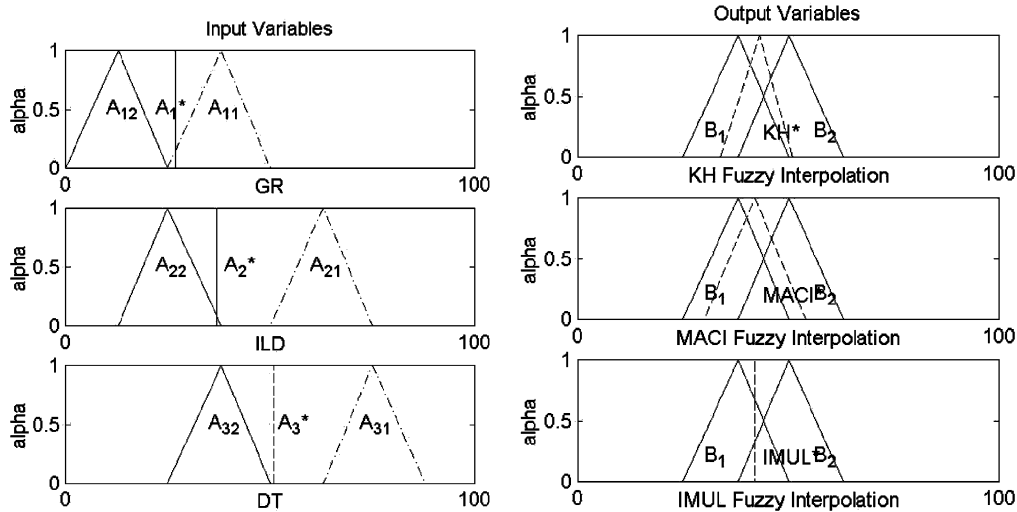


Fig. 9. Comparison of KH, MACI, and IMUL on petroleum data.

TABLE I  
COMPARISON RESULTS ON PETROLEUM DATA

Test case	Average MSE
Mamdani Fuzzy System without IMUL	0.0164
With KH*	0.0068
With MACI*	0.0056
With IMUL	0.0056

\* Obtained by means of COG defuzzification

system and the core PHI were used to calculate the average mean square errors. In the second test, we extracted the two input instances that could not find any fuzzy rules to perform inference and applied KH, MACI, and IMUL fuzzy rule interpolation techniques to interpolate PHI for them. Fig. 9 depicts a case when rule interpolation methods were used. Table I shows the comparison average mean square errors (MSE) of fuzzy systems constructed from the two tests.

We can observe that with the assistance of fuzzy rule interpolation techniques, the mean square error of the predicted PHI as compared to the core PHI has decreased quite significantly. This is partly due to the default prediction output for the two input instances being set to zero. The three fuzzy rule interpolation techniques yield similar MSE, where KHs error is somewhat higher than the ones obtained by IMUL and MACI.

Here we note again that in the case of IMUL, no defuzzification is needed because the observations are crisp. This is a clear advantage over MACI in real-world applications. When defuzzification comes into question, there is another decision: which method to choose. In the example, we experienced with COG, COA and MOM methods, and they produced different MSE errors. In Table I the best values are tabulated obtained with COG. Applying COG, MSE errors of IMUL and MACI are identical in the given examples, because  $B^*$  have the same reference point in both cases and  $B_{MACI}^*$  is a symmetrical fuzzy set.

In this case study, the number of input instances that cannot find any fuzzy rule to fire is small. In cases where more than half input instances in the prediction well cannot find any rule to fire, the Mamdani based fuzzy model could not be used for petrophysical properties prediction at all. With the applied fuzzy rule interpolation techniques (IMUL, MACI, or KH), the number of

fuzzy rules is considered the same, as no extra fuzzy rule is added into the system. However, the prediction ability has improved. This is a desirable characteristic for fuzzy petrophysical properties prediction, as an increase in the number of fuzzy rules would result in an increase in complexity that would make the examination of the fuzzy rule base more difficult.

### C. Applying in Hydrocyclone Parameter Determination

Hydrocyclones are used extensively in mineral processing and manufacturing industries to classify and separate particles suspended in fluid [46], [47]. Hydrocyclones normally have no moving parts. They are funnel-shaped devices manufactured in different dimensions to suit specific operations. A hydrocyclone is set up in a vertical arrangement with the slurry being fed into the hydrocyclone through the inlet. The bulk of the flow pattern will follow a downward spiral inside the cyclone along the wall containing coarse solid particles. The coarse particles leave the hydrocyclone through the underflow opening known as the spigot. On the other hand, an upward helical flow containing fine solid particles is discharged via the vortex finder as overflow. For a hydrocyclone of fixed geometry, the performance of the system depends on a number of parameters. Some of these parameters are fixed and some are variable. A number of these parameters are related to the physical size of the system, such as the internal diameters of the cyclone ( $D_c$ ), inlet pipe ( $D_i$ ), overflow pipe ( $D_o$ ) and spigot opening ( $D_u$ ). Other parameters are dependent on the operating conditions, the characteristics of the slurry, and the rate of feed into the hydrocyclone. The separation efficiency of particles of a particular size is determined by an operational parameter known as  $d50_c$ . This value indicates that 50% of a particular size particle is reported to the overflow and the other 50% to the underflow streams. The correct estimation of  $d50_c$  is important since it is directly related to the efficiency of operations and it will also enable control of the hydrocyclone.

In this case study, data collected from a Krebs hydrocyclone model D6B-120-839 has been used. There are a total of 70 training data and 69 testing data used in this study. The input parameters are inlet flow rate  $Q_i$ , density  $P_i$ , vortex finder height  $H$ , spigot opening  $D_u$ , and temperature of slurries  $T$  and the output is  $d50_c$ . When the set of sparse rules are used to perform control on the

TABLE II  
COMPARISON RESULTS ON HYDROCYCLONE DATA

Test case	Average MSE
Mamdani Fuzzy System without IMUL	149.72
With KH*	48.470
With MACI*	40.825
With IMUL	40.664

\* Obtained by means of COG defuzzification

testing data, 4 input instances cannot find any fuzzy rules to fire using the conventional fuzzy system. Similar to the previous application, from the observation and Euclidean distance measured on each input variable, the nearest fuzzy rules to the four input instances are determined for use by fuzzy rule interpolation.

Two tests are then designed similarly to the previous application. In the first test, the input instances are input to the Mamdani type fuzzy inference system [2]. The four inputs instances that find no fuzzy rules to fire are set to zero by default using the Mamdani type fuzzy inference system. In the second test, KH, MACI and IMUL fuzzy rule interpolation techniques are used for the four input instances that cannot find fuzzy rules to interpolate the predicted  $d50_c$ . Table II shows the comparison average MSE for the fuzzy systems constructed from the two tests.

From the average mean square errors of the two tests, it can be observed that the fuzzy hydrocyclone model performs unreasonably when no fuzzy rule interpolation technique is used. In contrast, when fuzzy rule interpolation techniques are applied, MSE is significantly better. This is mainly due to the four input instances that find no rule to fire and generate a default value of zero. Here, MSE of IMUL is even slightly better than that of MACI when applying the best defuzzification method for the given problem. With KH, MACI and IMUL fuzzy rule interpolation techniques, the number of fuzzy rules is not increased, but the prediction ability has improved. Again this has shown that IMUL and other fuzzy rule interpolation techniques play an important role to ensure the usability of a sparse fuzzy rule. IMUL has the advantage over the other two methods that it gives directly interpretable crisp output and it generates slightly better MSE.

## V. CONCLUSION

This paper has examined the problem of a sparse fuzzy rule base and insufficient training data that may cause undesirable prediction outcomes. This is mainly due to input instances that could not find any rule in the fuzzy rule base. To provide a solution to this problem, fuzzy rule interpolation techniques can be used. However, the majority of the fuzzy rule interpolation techniques published only present analysis limited to one input variable. This paper investigates KH, MACI, and an improved fuzzy interpolation technique for multidimensional input spaces. This technique can be used to interpolate the gaps between the rules for engineering problems with multidimensional input spaces. It also has the advantageous property that it does not require the application of any defuzzification methods when the observations are crisp, which is typical for engineering problems. This ensures that the set of sparse fuzzy rules generated by the fuzzy rule extraction technique will be usable in a practical system. This is significant as this will allow the use of a fuzzy system as an alternative for most engineering problems, at the same

time without increasing the number of fuzzy rules that allows more human control. Applications in the field of petroleum engineering and mineral processing have also been examined. The results have shown that this improved fuzzy rule interpolation technique for multidimensional input spaces can be used in engineering applications efficiently.

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